BIHARMONIC CURVES IN FOUR-DIMENSIONAL DAMEK-RICCI SPACES

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Abstract

Four-dimensional Damek-Ricci spaces are one-dimensional extensions of three-dimensional Heisenberg groups. In this note, we prove that there is no non-geodesic biharmonic curve in a four-dimensional Damek-Ricci space although such curves exist in three-dimensional Heisenberg groups.

1. Introduction

The notions of harmonic and biharmonic maps between Riemannian manifolds have been introduced by Eells and Sampson (see [5]).

For a map \( \phi : (M, g) \to (\overline{M}, \overline{g}) \) between Riemannian manifolds \((M, g)\) and \((\overline{M}, \overline{g})\), the energy functional \( E_1 \) is defined by

\[
E_1(\phi) = \frac{1}{2} \int_M \|d\phi\|^2 v_g.
\]

Critical points of \( E_1 \) are called harmonic maps, and are then solutions of the corresponding Euler-Lagrange equation

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\[ \tau_1(\phi) := \text{trace}\nabla^\phi d\phi = 0, \]

\(\nabla^\phi\) denotes the induced connection on the pull-back bundle \(\phi^{-1}(T\overline{M})\)
and \(\tau_1(\phi)\) is called the tension field of \(\phi\).

Biharmonic maps are the critical points of the bienergy functional

\[ E_2(\phi) = \frac{1}{2} \int_M \|\tau_1(\phi)\|^2 v_g, \]

whose Euler-Lagrange equation is given by the vanishing of the bitension field (see [6])

\[ \tau_2(\phi) := -\Delta^\phi \tau_1(\phi) - \text{trace} \overline{R}(d\phi, \tau_1(\phi))d\phi, \]

where \(\Delta^\phi = -\text{trace}_g (\nabla^\phi \nabla^\phi - \nabla^\phi)\) is the Laplacian on the sections of
\(f^{-1}(TN)\), and \(\overline{R}\) is the Riemannian curvature operator of \((\overline{M}, \overline{g})\).

Harmonic maps are obviously biharmonic and are absolute minimum
of the bienergy.

Nonminimal biharmonic submanifolds of the pseudo-euclidean space
and of the spheres have been studied in [2] and [4].

In [3], the authors studied non-geodesic biharmonic curves in the
three-dimensional Heisenberg group \(\mathbb{H}_3\), and gave an explicit description
of such curves in a Frenet frame.

Damek-Ricci spaces are some one-dimensional extensions of
Heisenberg groups. This class of manifolds has been introduced by
Damek and Ricci (see [1] for a large description of these spaces) to give
examples of harmonic manifolds that are not symmetric, and thus to
prove that, the conjecture of Lichnerowicz fails in the non-compact case.

By following the authors in [3], we construct an analogous Frenet
frame for four-dimensional Damek-Ricci spaces in which, we compute
explicitly the biharmonic equation for curves, and then we point out that
only geodesics are solutions of the equation.
In the next section, we recall briefly the geometry of the four-dimensional Damek-Ricci space, and in the last section, we give the four-dimensional Frenet frame in which, we compute the tension and the bitension fields, and we prove the non-existence result.

2. Damek-Ricci Space of Dimension 4

Let us consider the Lie bracket defined on the euclidean space $\mathbb{R}^4$ by:

\[
\begin{align*}
[X_0, Y_0] &= X_0, & [X_0, Z_0] &= 0, \\
[X_0, A_0] &= -\frac{1}{2} X_0, & [Y_0, Z_0] &= 0, \\
[Y_0, A_0] &= -\frac{1}{2} V_0, & [Z_0, A_0] &= -X_0,
\end{align*}
\]

where $(X_0, Y_0, Z_0, A_0)$ is the canonical basis of $\mathbb{R}^4$.

It is easy to check that $\mathbb{R}^4$ endowed with the Lie bracket defined above is a Lie algebra. We denote this Lie algebra by $\mathcal{D}_4 := (\mathbb{R}^4, [,])$.

The simply connected Lie group $\mathbb{D}_4$ of the Lie algebra $\mathcal{D}_4$ with the induced left invariant metric $g$ is called Damek-Ricci space of dimension 4.

Note that the three-dimensional euclidean space $\mathbb{R}^3 \equiv \mathbb{R}^3 \times \{0\}$ with the restriction of the Lie bracket given above defines a Lie algebra, whose simply connected Lie group with its left invariant metric is the three-dimensional Heisenberg group $\mathbb{H}_3$.

In general, Damek-Ricci spaces are some one-dimensional extensions of Heisenberg groups.

Denote by $(x, y, z, t)$, the global coordinates system on $\mathbb{R}^4$. The canonical basis $(X_0, Y_0, Z_0, A_0)$ induced on $\mathcal{D}_4$, a left invariant basis $(X, Y, Z, A)$ given by:
The following important geometric properties hold for Damek-Ricci spaces (see [1]):

**Proposition 2.1.** A Damek-Ricci space is an Einstein manifold and a Hadamard manifold; that is of nonpositive sectional curvature.

It is maybe important to notify that contrary to Damek-Ricci spaces, the sectional curvatures of Heisenberg groups attain both positive and negative values.

### 3. Biharmonicity of Curves in $(\mathbb{D}_4, g)$

#### 3.1. Frenet frame and biharmonic equation

Let $\gamma : I \to (\mathbb{D}_4, g)$ be a regular curve in the four-dimensional Damek-Ricci space parametrized by the arclength.

Denote by $T = \gamma'$, the tangent unit vector field along $\gamma$. 
We have the following result:

**Lemma 3.1.** There are vector fields $N, B, \Lambda$ along the curve $\gamma$ and some functions $\kappa, \tau,$ and $\theta$ defined on $\gamma(I) \subset \mathbb{D}_4$ such that:

\[
\begin{align*}
\nabla_T T &= \kappa N, \\
\nabla_T N &= -\kappa T + \tau B, \\
\nabla_T B &= -\tau N + \theta \Lambda, \\
\n\nabla_T \Lambda &= -\theta B.
\end{align*}
\]

**Proof.** (a) We have $g(\nabla_T T, T) = 0$. Then, there exists a function $\kappa \in C^\infty(I)$ and a unitary vector field $N \in \gamma^{-1}(T\mathbb{D}_4)$, orthogonal to $T$ such that $\nabla_T T = \kappa N$.

(b) Next $g(N, N) = 1$ implies $g(\nabla_T N, N) = 0$, and from the equality $g(N, T) = 0$, we derive the relation

\[
g(\nabla_T N, T) + g(N, \nabla_T T) = g(\nabla_T N, T) + \kappa
\]

\[
= g(\nabla_T N + \kappa T, T)
\]

\[
= 0.
\]

Hence $\nabla_T N + \kappa T \in (\text{span} \{T, N\})^\perp$, and it exists a smooth function $\tau$ on $\gamma(I)$ and a unitary vector field $B \in \gamma^{-1}(T\mathbb{D}_4)$ such that the system \{T, N, B\} is orthonormal and $\nabla_T N + \kappa T = \tau B$. This gives the second relation of the system.

Similarly, $g(B, B) = 1$ leads to $g(\nabla_T B, B) = 0$ and from the relation $g(B, T) = 0$, we get

\[
g(\nabla_T B, T) + g(B, \nabla_T T) = g(\nabla_T B, T) + g(B, \kappa N)
\]

\[
= g(\nabla_T B, T)
\]

\[
= 0.
\]
From \( g(B, N) = 0 \), we have
\[
g(\nabla_T B, N) + g(B, \nabla_T N) = g(\nabla_T B, N) + \tau
= g(\nabla_T B + \tau N, N)
= 0.
\]
Hence \( \nabla_T B + \tau N \in (\text{span } \{T, N, B\})^\perp \), and it exists a smooth function \( \theta \) on \( \gamma(I) \), and a unitary vector field \( \Lambda \in \gamma^{-1}(T \mathbb{R}_4) \) such that the system \( \{T, N, B, \Lambda\} \) is orthonormal and \( \nabla_T B + \tau N = 0 \Lambda \). So, we obtain the third equation of the system.

Furthermore, from \( g(\Lambda, \Lambda) = 1 \), we have \( g(\nabla_T \Lambda, \Lambda) = 0 \) and the relation \( g(\Lambda, T) = 0 \) leads to
\[
g(\nabla_T \Lambda, T) + g(\Lambda, \nabla_T T) = g(\nabla_T \Lambda, T) = 0.
\]
Also from \( g(\Lambda, N) = 0 \), it follows that
\[
g(\nabla_T \Lambda, N) + g(\Lambda, \nabla_T N) = g(\nabla_T \Lambda, N) = 0,
\]
and the relation \( g(\Lambda, B) = 0 \) gives
\[
g(\nabla_T \Lambda, B) + g(\Lambda, \nabla_T B) = g(\nabla_T \Lambda, B) + \theta = 0.
\]
We get then \( g(\nabla_T \Lambda, B) = -\theta \) and the last relation of the system follows.

The functions \( \kappa, \tau, \) and \( \theta \) defined in the Lemma 3.1 are the curvature, the torsion, and the bitorsion along the curve \( \gamma \), respectively.

The quadruplet \( \{T, N, B, \Lambda\} \) is called the \textit{four-dimensional Frenet frame}.

The tension field \( \tau_1(\gamma) \) and the bitension field \( \tau_2(\gamma) \) of the curve \( \gamma \) are then given in the four-dimensional Frenet frame \( \{T, N, B, \Lambda\} \) by:
Proposition 3.1.

\[ \tau_1(\gamma) = \kappa N, \]

and

\[ \tau_2(\gamma) = -\frac{3}{2} \left( \kappa^2 \right)' T + \left( \kappa^* - \kappa^3 - \kappa \tau^2 \right) N \]

\[ + \left( \kappa' \tau + (\kappa') T \right) B + \kappa \tau \partial - \kappa R(T, N) T, \]

where \( R \) denotes the Riemann curvature of \( g \).

Proof. From the definition of the tension field, we get directly \( \tau_1(\gamma) = \nabla_T T \) and the formula follows.

We have \( \Delta^2 \tau_1(\gamma) = \nabla_T \nabla_T \tau_1(\gamma) \).

It follows that

\[ \tau_2(\gamma) = \nabla_T \nabla_T \tau_1(\gamma) - R(T, \tau_1(\gamma)) T, \]

\[ = \nabla_T \nabla_T (\nabla_T T) - R(T, \nabla_T T) T, \]

\[ = \nabla_T \nabla_T (\kappa N) - R(T, \kappa N) T, \]

\[ = \nabla_T (\kappa' N + \kappa \nabla_T N) - \kappa R(T, N) T, \]

\[ = \nabla_T (\kappa' N + \kappa \nabla_T N) - \kappa R(T, N) T, \]

\[ = \nabla_T (\kappa' N - \kappa^2 T + \kappa \tau B) - \kappa R(T, N) T, \]

\[ = \kappa' N + \kappa' \nabla_T N - \left( \kappa^2 \right)' T - \left( \kappa^2 \right) \nabla_T T \]

\[ + (\kappa') B + \kappa \tau \b T - \kappa R(T, N) T, \]

\[ = \kappa' N + \kappa' (-\kappa T + \tau B) - \left( \kappa^2 \right)' T \]

\[ - \left( \kappa^2 \right) (\kappa N) + (\kappa') B \]

\[ + \kappa \tau (-\tau N + \partial) - \kappa R(T, N) T. \]
3.2. The non-existence result

From Proposition 2.1 and by using the formulas of the tension field $\tau_1(\gamma)$ and of the bitension field $\tau_2(\gamma)$ computed above, it follows:

**Theorem 3.1.** There is no non-geodesic biharmonic curve in $(\mathbb{D}_4, g)$.

**Proof.** Let $\gamma : I \to (\mathbb{D}_4, g)$ be a regular curve parametrized by the arclength and $\gamma, (T, N, B, \Lambda)$, the four-dimensional Frenet frame along $\gamma$.

The curve $\gamma$ is non-geodesic and biharmonic, if and only if

$$\tau_1(\gamma) \neq 0 \text{ and } \tau_2(\gamma) = 0.$$  

Denote by $\sigma$ the sectional curvature of $(\mathbb{D}_4, g)$, we have

$$g(\tau_1(\gamma), N) = \kappa,$$

$$g(\tau_2(\gamma), T) = -\frac{2}{3} (\kappa^2)' - \kappa g(R(T, N)T, T),$$

$$= -\frac{2}{3} (k^2),$$

and

$$g(\tau_2(\gamma), N) = (\kappa^* - \kappa^3 - \kappa \tau^2) - \kappa g(R(T, N)T, N),$$

$$= \kappa^* - \kappa^3 - \kappa \tau^2 + \kappa \sigma(T, N).$$

If the curve $\gamma$ is non-geodesic and biharmonic, then we should have

$$\begin{cases} \kappa \neq 0, \\ g(\tau_2(\gamma), T) = 0, \\ g(\tau_2(\gamma), N) = 0. \end{cases}$$

That is,

$$\begin{cases} \kappa = \text{constant} \neq 0, \\ \kappa^2 + \tau^2 = \sigma(T, N). \end{cases}$$

Since $(\mathbb{D}_4, \bar{g})$ is a Hadamard space, its sectional curvature $\sigma$ is nonpositive; thus, we get a contradiction. \qed
References


